

On Quasinormal Modes for Scalar Perturbations of Static Spherically Symmetric Black Holes in Nash Embedding Framework

Sergio C. Ulhoa*

Instituto de Física, Universidade de Brasília, 70910-900, Brasília, DF, Brazil.

Ronni G. G. Amorim†

*Faculdade Gama, Universidade de Brasília,
72444-240, Setor Leste (Gama), Brasília, DF, Brazil.*

Abraão J. S. Capistrano‡

*Federal University of Latin-American Integration,
Casimiro Montenegro Filho Astronomy Center,
Itaipu Technological Park, 85867-970, P.o.b: 2123,
Foz do Iguassu, PR, Foz do Iguaçu-PR, Brazil*

Abstract

In this paper we investigate scalar perturbations of black holes embedded in a five dimensional bulk space. It is calculated the quasinormal frequencies of a such black holes using the third order of Wentzel, Kramers, Brillouin (WKB) approximation for scalar perturbations. The results are presented in tables along the text.

PACS numbers: 04.20-q; 04.20.Cv; 02.20.Sv

Keywords: Quasinormal frequencies; scalar perturbations.

* sc.ulhoa@gmail.com

† ronniamorim@gmail.com

‡ abraao.capistrano@unila.edu.br

I. INTRODUCTION

The black hole solutions can be perturbed by some matter field, then it leads to oscillations, which can be explained by an analogy with a vibrating string, in such a system the oscillations decrease with time as the string loses energy to the environment [1]. Similarly to the string, where its frequency can be expressed by a complex parameter called quasi-normal frequency, the perturbations of black holes evolve with time as an oscillation with complex frequency, in which the real part is responsible for the period of oscillation and the imaginary part is responsible for damping or amplification. Thus proper oscillations of black holes are also called quasi-normal modes.

Historically, the theoretical study about perturbation of black holes begins with the seminal paper of Regge and Wheeler in 1957, in which they studied the stability of Schwarzschild black hole submitted to small perturbations [2]. In the 1970's, the quasi-normal modes were used by Vishveshwara, in calculations of scattering of gravitational waves by a Schwarzschild black hole [3]. Since then, the study of this issue were directed to several problems. Black holes stability is an example of this importance. The relevance of studying the stability of these astrophysical objects is the possibility to understand theories in higher dimensions as aspects of brane-world and string theory[4, 5] and variants. Since the uniqueness theorem is not applicable to more than four dimensions, stability can be the criterion to select physical solutions [6, 7]. Another important application of the quasi-normal modes of black holes is in the study of the ADS/CFT correspondence [8], whereby the imaginary part of the quasi-normal fundamental frequency describe the thermalization time in a conformal invariant field theory in the border of the anti de-Sitter space-time. Recently, the quasi-normal modes have been related to quantum gravity; some evidences show that highly damped quasi-normal modes can be important in the attempts to quantize the area of black holes [9]. In addition to the advantages presented, there are prospects that in the near future oscillations of astrophysical black holes will be observed using gravitational waves detectors [10]. This fact is relevant in the sense that quasi-normal modes carry information about stellar objects, such as mass, charge and angular momentum.

Considering the relevance of the quasi-normal modes, in this work we study scalar perturbation of a black hole embedded in a five-dimensional bulk space by calculating its quasi-normal frequencies. In this sense, this paper is organized as follows. In section 2, we use the

Nash embedding theorem to study the induced four dimensional equations embedded in a five-dimensional bulk space. In section 3, we present the induced potential in a embedded spherically symmetric vacuum solution. In section 4, we discuss about scalar perturbations. In section 5, we present the quasi-normal modes for treated system using WKB approach and using this result we discuss about black hole stability. In section 6, we present our concluding remarks.

II. EMBEDDED INDUCED FOUR DIMENSIONAL EQUATIONS

The recent proof of the Poincaré conjecture by G. Perelman [11, 12] suggests a new paradigm for geometry and in particular for Einstein's gravitation and cosmology, namely the possibility that we can deform space-times in arbitrary directions, changing its shape. It has its origins from a solution for the Riemann curvature ambiguity was conjectured by L. Schläfli in 1871, proposing that the Riemann manifolds should be embedded in a larger bulk space, such that its Riemann curvature \mathcal{R} would act as a curvature reference for all embedded manifolds, just like the flat Euclidean space \mathbb{R}^3 acts as a curvature reference for surfaces. The Schläfli conjecture is the origin of the embedding problem for Riemann's geometry. Formally we may write his proposal as

$$\mathcal{R}(U, V)W = R(U, V)W + \text{extrinsic terms} \quad (1)$$

To detail, we must re-write this expression in terms of components in a suitable reference frame. This can be done in an arbitrary vielbein defined in the bulk, and then separating the bulk curvature tensor in normal and tangent components. With the purpose of understanding the meaning of the Schläfli proposal we write these components in the Gaussian frame defined by the embedding map itself. In this frame the components of the extrinsic terms appear as the extrinsic curvature component. This allows us to relate different geometries with different properties through a differential embedding. In this paper, we explore the consequences of such embedding using the dynamics properties of the extrinsic curvature. In order to make an appropriate embedding between smooth (differentiable) geometries, we use the Nash embedding theorem [13] in order to propose a new theoretical structure able to relate it to a physical theory.

Nash showed that any embedded perturbed metric $\tilde{g}_{\mu\nu}$ can be generated by a continuous

sequence of small metric perturbations of a given initially unperturbed geometry $g_{\mu\nu}$ by means of

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta y^a k_{\mu\nu a} + \delta y^a \delta y^b g^{\rho\sigma} k_{\mu\rho a} k_{\nu\sigma b} + \cdots, \quad (2)$$

or, equivalently,

$$k_{\mu\nu} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y}, \quad (3)$$

where y is the coordinate related to extra dimensions. Since Nash's smooth deformations are applied to the embedding process, the coordinate y , usually noticed in rigid embedded models, e.g [14, 15], can be omitted in the process for perturbing the element line. This seems particularly interesting to astrophysical and cosmological proposes in which traditionally the gravitational perturbation mechanisms are essentially plagued by coordinate gauges due to the group of diffeomorphisms.

The Einstein-Hilbert principle leads to D-dimensional Einstein's equations for the bulk metric \mathcal{G}_{AB} in arbitrary coordinates

$$\mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} \mathcal{G}_{AB} = G_* T_{AB}^*, \quad (4)$$

where we have dispensed the bulk cosmological constant and T_{AB}^* denotes the energy-momentum tensor of the known matter and gauge fields. The constant G_* determines the D-dimensional energy scale. For the present application, capital Latin indices run from 1 to 5. Small case Latin indices refer to the only one extra dimension considered. All Greek indices refer to the embedded space-time counting from 1 to 4.

Concerning the confinement, the four-dimensionality of the space-time is an experimentally established fact associated with the Poincaré invariance of Maxwell's equations and their dualities, also valid for Yang-Mills gauge fields restricted to four-dimensions [16, 17]. Even though the duality properties can be mathematically extended to higher dimensions, we adopt it as a condition based on experimental backgrounds [18]. Therefore, all matter, which interacts with these gauge fields, must for consistency be also defined in the four-dimensional space-times. This consideration complements a physical interpretation for the Nash theorem that provides an interesting mechanism for perturbing and creating new geometries. In this five-dimensional framework present here, only the components that access higher dimensions are related to the extrinsic curvature $k_{\mu\nu}$, while the metric components are confined to the geometry itself, as shown in Eq.(2). On the other hand, as well known, in spite of all efforts

made so far, the gravitational interaction has failed to fit into a similar gauge scheme, so that the gravitational field does not necessarily have the same four-dimensional limitations, regardless the location of its sources.

In order to recover Einstein's gravity by reversing the embedding, the confinement of ordinary matter and gauge fields implies that the tangent components of $G_*T_{AB}^*$ in the above equations must coincide with $(8\pi GT_{\mu\nu})$ where $T_{\mu\nu}$ is the energy-momentum tensor of the confined sources. As it may have been already noted, we are essentially reproducing a framework similar to brane-world program [26], with the difference that we apply a dynamical differential embedding and has nothing to do with branes as those defined in string/M theory. Using the Nash embedding theorem together with the four-dimensionality of gauge fields, one can obtain the Einstein-Hilbert principle for the bulk and a D-dimensional energy scale G_* .

In addition, one can define a five-dimensional local embedding with an embedding map $\mathcal{Z} : V_4 \rightarrow V_5$. We admit that \mathcal{Z}^μ is a regular and differentiable map with V_4 and V_5 being the four embedded space-time and the bulk, respectively. The components $\mathcal{Z}^A = f^A(x^1, \dots, x^4)$ associate with each point of V_4 a point in V_5 with coordinates \mathcal{Z}^A . These coordinates are the components of the tangent vectors of V_4 . Accordingly, calculating the components of Eq.(4) one can find the induces equations for the embedded geometry

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - Q_{\mu\nu} = -8\pi GT_{\mu\nu} \quad (5)$$

$$k_{\mu;\rho}^\rho - h_{,\mu} = 0, \quad (6)$$

where now $T_{\mu\nu}$ is the energy-momentum tensor of the confined matter. The quantities $h^2 = h.h$ with $h = g^{\mu\nu}k_{\mu\nu}$ and $K^2 = k^{\mu\nu}k_{\mu\nu}$ are the mean curvature and Gaussian curvature, respectively. Moreover, one defines

$$Q_{\mu\nu} = k_{\mu}^{\rho}k_{\rho\nu} - hk_{\mu\nu} - \frac{1}{2}(K^2 - h^2)g_{\mu\nu}. \quad (7)$$

This tensor is independently conserved, as it can be directly verified that (semicolon denoting covariant derivative with respect to $g_{\mu\nu}$)

$$Q^{\mu\nu}{}_{;\nu} = 0. \quad (8)$$

A detailed derivation of these equations can be found in [20–22] and references therein as well the higher dimensional case. Hereafter, we use a system of unit such that $c = G = 1$.

III. INDUCED POTENTIAL IN A SPHERICALLY SYMMETRIC VACUUM SOLUTION

As shown in a previous work [19], we start with the general static spherically symmetric induced metric that can be described by the line element as

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \quad (9)$$

where we denote the functions $A(r) = A$ and $B(r) = B$. Thus, one can obtain the following components for the Ricci tensor:

$$\begin{aligned} R_{rr} &= \frac{B''}{2B} - \frac{1}{4} \frac{B'}{B} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{A'}{A} \\ R_{\theta\theta} &= -1 + \frac{r}{2A} \left(-\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} \\ R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \\ R_{tt} &= -\frac{B''}{2A} + \frac{1}{4} \frac{B'}{A} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{B'}{A} \end{aligned}$$

where we have $\frac{dA}{dr} = A'$ e $\frac{dB}{dr} = B'$.

From Eq.(5), the gravitational-tensor vacuum equations (with $T_{\mu\nu} = 0$) can be written in alternative form as

$$R_{\mu\nu} + \frac{1}{2}Qg_{\mu\nu} = Q_{\mu\nu} \quad (10)$$

where we use the contraction $Q = g^{\mu\nu}Q_{\mu\nu}$.

The general solution of Codazzi equations Eq.(6) is given by

$$k_{\mu\nu} = f_\mu g_{\mu\nu} \quad (\text{no sum on } \mu), \quad (11)$$

Taking the former equation and the definition of $Q_{\mu\nu}$, one can write

$$Q_{\mu\nu} = f_\mu^2 g_{\mu\nu} - \left(\sum_\alpha f_\alpha \right) f_\mu g_{\mu\nu} - \frac{1}{2} \left(\sum_\alpha f_\alpha^2 - \left(\sum_\alpha f_\alpha \right)^2 \right) g_{\mu\nu},$$

where

$$U_\mu = f_\mu^2 - \left(\sum_\alpha f_\alpha \right) f_\mu - \frac{1}{2} \left(\sum_\alpha f_\alpha^2 - \left(\sum_\alpha f_\alpha \right)^2 \right) \delta_\mu^\mu.$$

Consequently, we can write $Q_{\mu\nu}$ in terms of f_μ as

$$Q_{\mu\nu} = U_\mu g_{\mu\nu} \quad (\text{no sum on } \mu). \quad (12)$$

A straightforward consequence of the homogeneity of Codazzi equations Eq.(6) in five-dimensions is that the individual arbitrariness of the functions f_μ can be reduced to a unique arbitrary function α that depends on the radial coordinate. Hence, the equation (11) turns to be

$$k_{\mu\nu} = \alpha(r)g_{\mu\nu} , \quad (13)$$

with $\alpha(r) = \alpha$. With a straightforward calculation of the former equations, one can obtain the coefficients of the metric as

$$B(r) = 1 + \frac{K}{r} + \frac{9}{r} \int \alpha^2(r)r^2 dr , \quad (14)$$

and

$$A(r) = [B(r)]^{-1} = \left[1 + \frac{k}{r} + \frac{9}{r} \int \alpha^2(r)r^2 dr \right]^{-1} , \quad (15)$$

where k is a constant. In order to constrain this arbitrariness, we look at the characteristics of the extrinsic curvature itself in the asymptotic limit, which will be important to an astrophysical application. The extrinsic curvature at infinity goes to a flat space obeying the asymptotically conformal flat condition. This can be understood as the following form

$$\lim_{r \rightarrow \infty} k_{\mu\nu} = \lim_{r \rightarrow \infty} \alpha(r) \lim_{r \rightarrow \infty} g_{\mu\nu} . \quad (16)$$

Since the function $\alpha(r)$ must be analytical at infinity, one can write the simplest option

$$\alpha(r) = \sum_{n=i}^s \frac{\sqrt{-\alpha_0}}{\gamma^* r^n} , \quad (17)$$

where the sum is upon all scalar potentials and the indices i and s are real numbers. Since these scalar potentials have its origin in the extrinsic curvature they do not remain confined in the embedded geometry propagating in the extra-dimension. The index n represents all the set of scalar fields that fall off with r -coordinate following the inverse n^{th} power law. The equation (17) is essentially the representation of the effect of extrinsic curvature leading to a local modification of the space-time without producing umbilical point as expected for a spherical geometry. Depending on the variation of the function $\alpha(r)$ one can have a bent or stretched geometry without ripping off the manifold, and, curiously, in the same notion as pointed by Riemann himself [34]. It was shown that the parameter α_0 has cosmological magnitude [19, 23], i.e., it does not depend on individual astrophysical properties and has the same units as the Hubble constant. Its modulus is of the order of $0.677 \text{ km.s}^{-1}.\text{Mpc}^{-1}$.

In addition, to keep the right dimension of Eq.(17) we have introduced a unitary parameter γ^* that has the inverse unit of Hubble constant and also establishes the cosmological horizon in Eq.(9). It is important to stress that the form of the function $\alpha(r)$ could be no other since the relativistic effects in astrophysical scale is estimated to be 10^{-8} times weaker than Newtonian ones [24] and it must obey a distance decaying law ($\sim 1/r$) or their smooth deviations ($\sim 1/r^2, \sim 1/r^3$).

Using Eqs.(14) and (17), one can obtain an explicit form of the coefficient $B(r)$ given by

$$B(r) = 1 + \frac{k(9\alpha_0^2 + 1)}{r} - \sum_{n=i}^s \frac{9\alpha_0}{\gamma^*(2n-3)} r^{2(1-n)}. \quad (18)$$

In terms on the correspondence principle with Einstein gravity, we set $k(9\alpha_0^2 + 1) = -2M$, which remains valid even in the limit when $\alpha_0 \rightarrow 0$ in order to obtain the asymptotically flat solution.

IV. SCALAR PERTURBATIONS

In this section, we focus our attention on the study of Eq.(18) under scalar perturbations. The study of quasinormal modes can be treated by scalar fields and appropriate metrics. In this section, we construct the Klein-Gordon equation in a curved space-time. In order to achieve such a goal, we consider the line element in a form

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 + r^2d\Omega. \quad (19)$$

It is well known that the massless Klein-Gordon equation is given by

$$\nabla_\mu \nabla^\mu \Phi = 0, \quad (20)$$

or, equivalently

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) = 0, \quad (21)$$

where $\Phi = \Phi(r, \theta, \phi, t)$. Using Eq.(19), we can write

$$\sqrt{-g} = r^2 \sin \theta. \quad (22)$$

Moreover, Eq.(21) can be written as

$$\frac{1}{f(r)} \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial}{\partial r} \left(f(r) \frac{\partial \Phi}{\partial r} \right) - \frac{2f(r)}{r} \frac{\partial \Phi}{\partial r} - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \quad (23)$$

Taking the ansatz

$$\Phi(r, \theta, \phi, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{l,m}(\theta, \phi) e^{-i\omega t} \frac{\psi_{l,m}(r)}{r}, \quad (24)$$

we obtain

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \right] Y_{l,m}(\theta, \phi) = -l(l+1) Y_{l,m}(\theta, \phi), \quad (25)$$

where $Y_{l,m}(\theta, \phi)$ are the spherical harmonics.

$$\left(\frac{-\omega^2}{f(r)} + \frac{l(l+1)}{r^2} - f(r) \frac{\partial^2}{\partial r^2} - f'(r) \frac{\partial}{\partial r} - \frac{2f(r)}{r} \frac{\partial}{\partial r} \right) \frac{\psi_{l,m}(r)}{r} = 0. \quad (26)$$

Taking the following coordinates change $dx = dr/f(r)$, we obtain

$$\frac{d^2 \psi_{l,m}(r)}{dx^2} + (\omega^2 - V(x)) \psi_{l,m}(r) = 0, \quad (27)$$

where we denote

$$V(x) = f(x) \frac{l(l+1)}{x^2} + f(x) \frac{f'(x)}{x}.$$

If we consider equation (18), then we finally get

$$f(r) = 1 - \frac{2M}{r} + \sum_j \frac{\alpha_0}{2j-3} r^{2(1-j)}. \quad (28)$$

Thus once the potential V is well defined, we can plot its behaviour for different values of α and l , which are given in fig.(1) and fig.(2).

For a massive field the potential is modified to $V(r) \rightarrow V(r) + \mu^2 f(r)/r$, the angular part remains the same. In fig.(3) we plot such a potential for $\alpha = 0.677$, $M = 1$ and $\mu = 0.1M$.

V. QUASINORMAL MODES IN WKB APPROXIMATION

In this section, we use the Wentzel, Kramers, Brillouin (WKB) approximation in such a regime the frequency is given by [36]

$$\frac{i(\omega^2 - V_0)}{\sqrt{-2V_0''}} - \sum_{i=2}^k \Lambda_i = n + \frac{1}{2},$$

where V_0 and V_0'' are the effective potential and its second derivative respectively, taken at the point of the maximum of V . The sum above represents high order correction in the usual WKB method [37]. Thus the higher is the orders of approximation the better is the

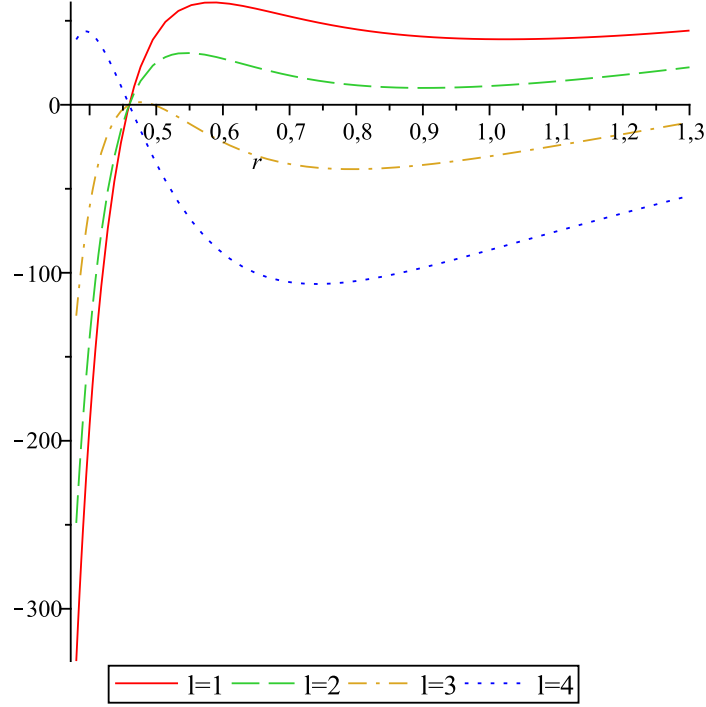


FIG. 1. The potential V in function of the radius for values $l = 1$ (solid thick line), $l = 2$ (long-dashed line), $l = 3$ (short-dashed line) and $l = 4$ (solid thin line) for the fixed values of $\alpha = 0.677$, $M = 1$ and $\mu = 0$.

result, which could be comparable to numerical methods [36]. In this sense it is enough to work with WKB approximation of third order [38], which is given by

$$\omega_{n,l}^2 = [V_0 + (-2V_0'')^{1/2}\Lambda] - i \left(n + \frac{1}{2} \right) (-2V_0'')^{1/2}(1 + \Omega), \quad (29)$$

where

$$\Lambda = \frac{1}{(-2V_0'')^{1/2}} \left[\frac{1}{8} \left(\frac{V_0^{(4)}}{V_0''} \right) \left(\frac{1}{4} + \beta^2 \right) - \frac{1}{288} \left(\frac{V_0'''}{V_0''} \right)^2 (7 + 60\beta^2) \right],$$

and

$$\begin{aligned} \Omega = \frac{1}{(-2V_0'')} & \left[\frac{5}{6912} \left(\frac{V_0'''}{V_0''} \right)^4 (77 + 188\beta^2) - \frac{1}{384} \left(\frac{(V_0''')^2 V_0^{(4)}}{(V_0'')^3} \right) (51 + 100\beta^2) + \right. \\ & \left. + \frac{1}{2304} \left(\frac{V_0^{(4)}}{V_0''} \right)^2 (67 + 68\beta^2) + \frac{1}{288} \left(\frac{V_0''' V_0^{(5)}}{(V_0'')^2} \right) (19 + 28\beta^2) - \frac{1}{288} \left(\frac{V_0^{(6)}}{V_0''} \right) (5 + 4\beta^2) \right], \end{aligned}$$

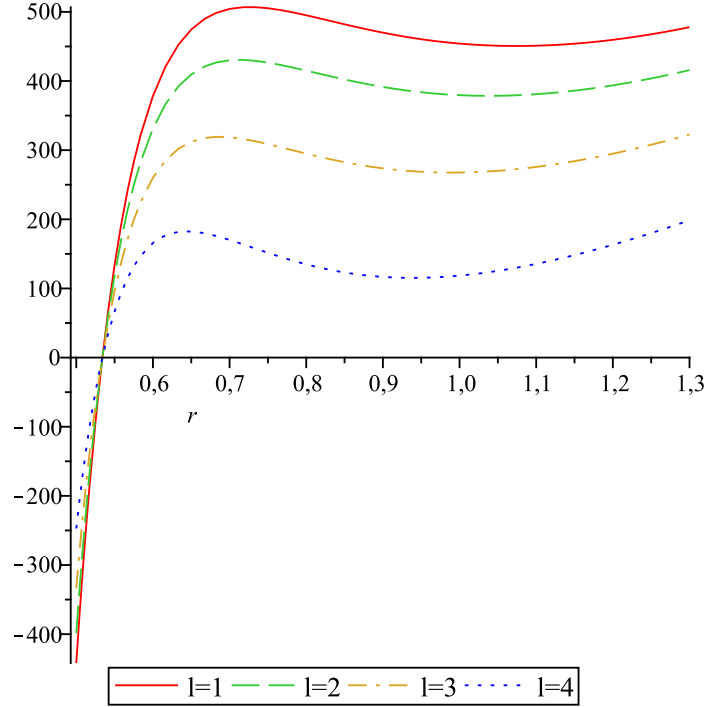


FIG. 2. The potential V in function of the radius for values $l = 1$ (solid thick line), $l = 2$ (long-dashed line), $l = 3$ (short-dashed line) and $l = 4$ (solid thin line) for the fixed values of $\alpha = 2$, $M = 1$ and $\mu = 0$.

Here $\beta = n + \frac{1}{2}$ and $V_0^{(n)} = \frac{d^n V}{dx^n}|_{x=x_0}$. The point x_0 is the solution of the equation $\frac{dV}{dx}(x_0) = 0$, which means a point of maximum of the effective potential. We point out that x is the tortoise coordinate, already defined.

Next we chart our results on tables IV, V and VI. We see from IV that the scalar perturbation for a realistic value of α is not stable. This complements our previous work in which we concluded that such a solution is not globally thermodynamically stable and their stability is reduced to very constrained range[35]. On the other hand from tables V and VI we see that the stability is achieved by a choice of a hypothetical value of α (in this case, $\alpha > 0.667$ will lead to a more faster accelerated regime as expected for a phantom-like cosmology [23]) or for a massive scalar field. In all scenarios the real part of the frequency increases rapidly, which indicates that the quasi-normal frequency oscillates faster for each mode. Hence, due to the positive growing of the imaginary part, such a behaviour is very similar to a resonance.

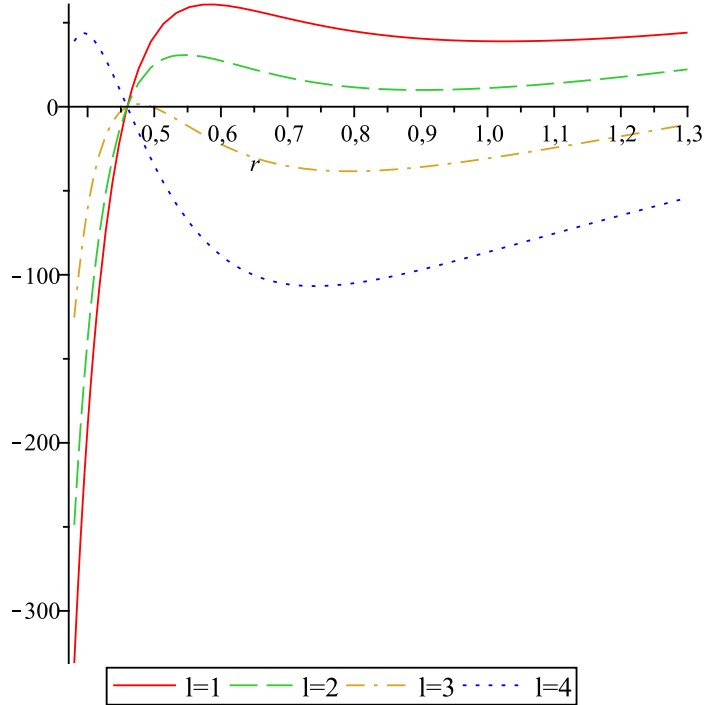


FIG. 3. The potential V in function of the radius for values $l = 1$ (solid thick line), $l = 2$ (long-dashed line), $l = 3$ (short-dashed line) and $l = 4$ (solid thin line) for the fixed values of $\alpha = 0.677$, $M = 1$ and $\mu = 0.1M$.

VI. CONCLUSION

In this article we have analyzed the stability of the line element obtained in [19, 35], under scalar perturbations. Such a metric was derived by searching for a solution with four-dimensional metric spherically symmetric in a five dimensional bulk space. Thus we identified an effective potential that composes the gravitational field equations. We have calculated the quasi-normal modes of Klein-Gordon equation in order to analyze the stability of our metric. We concluded that the massive scalar perturbation can yields a stable configuration or either a non realistic massless perturbation, as it is possible to see from tables V and VI. We point out that the instability of the solution for massless scalar perturbation with $\alpha = 0.677$ complements our previous observation about the thermodynamical

TABLE I. Quasinormal modes of scalar perturbations for $\alpha = 0.677$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$4.444548318 - 8.863420102i$	4	0	$2.789319410 + 26.91329506i$
	1	$11.59600413 - 26.25141968i$	1	1	$136.9620887 + 165.7373689i$
2	0	$2.556070877 - 9.893346520i$	2	1	$317.9204012 + 354.6421808i$
	1	$6.342841226 - 24.95915206i$	3	1	$540.1554148 + 583.4781202i$
	2	$12.94011467 - 41.62859430i$	5	0	$6.400241451 + 55.35392070i$
3	0	$0.3142845589 + 12.96051935i$	1	2	$301.0022385 + 362.7954640i$
	1	$38.84486810 - 53.32057992i$	2	2	$697.9325995 + 777.2504936i$
	2	$93.40199364 - 111.3133472i$	3	2	$1185.464821 + 1279.187469i$
	3	$160.5504266 - 181.4993689i$	4	2	$1749.721283 + 1855.936081i$

TABLE II. Quasinormal modes of scalar perturbations for $\alpha = 2$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$28.98497167 - 36.65093824i$	4	0	$14.72079936 - 30.28374333i$
	1	$103.8128126 - 126.3594331i$	1	1	$47.53015516 - 80.58865151i$
2	0	$29.20727668 - 38.11077359i$	2	1	$100.4540639 - 146.2752812i$
	1	$106.1438323 - 128.6137680i$	3	1	$169.1566913 - 224.7764688i$
	2	$224.2314700 - 254.7665296i$	5	0	$.5248454700 + 28.40271014i$
3	0	$26.35460874 - 37.16001393i$	1	2	$45.08904231 - 85.15794051i$
	1	$95.10645314 - 118.6073793i$	2	2	$116.4495623 - 166.1450658i$
	2	$200.6518286 - 232.2842502i$	3	2	$205.7296676 - 263.8287746i$
	3	$333.1889473 - 371.1040372i$	4	2	$309.9693513 - 375.5150908i$

features of such a solution.

[1] E. Berti, V. Cardoso, J.P.S Lemos, Phys. Rev. D, 70, 12, 124006,(2004).

TABLE III. Quasinormal modes of massive scalar perturbations for $\alpha = 0.677$ and $\mu = 0.1M$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$29.00936098 - 36.67332200i$	4	0	$14.74857813 - 30.30037176i$
	1	$103.9107237 - 126.4501766i$	1		$47.63840864 - 80.66160380i$
2	0	$29.23683246 - 38.13734008i$	2		$100.6803470 - 146.4444515i$
	1	$106.2610002 - 128.7222323i$	3		$169.5250565 - 225.0711954i$
	2	$224.4751039 - 254.9976125i$	5	0	$0.5217309759 + 28.39953701i$
3	0	$26.38849620 - 37.18864399i$	1		$45.02864470 - 85.11181068i$
	1	$95.24043132 - 118.7279173i$	2		$116.3098696 - 166.0264653i$
	2	$200.9301766 - 232.5433005i$	3		$205.4963229 - 263.6210918i$
	3	$333.6438082 - 371.5352605i$	4		$309.6292522 - 375.2043911i$

TABLE IV. Quasinormal modes of scalar perturbations for $\alpha = 0.677$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$4.444548318 - 8.863420102i$	4	0	$2.789319410 + 26.91329506i$
	1	$11.59600413 - 26.25141968i$	1		$136.9620887 + 165.7373689i$
2	0	$2.556070877 - 9.893346520i$	2		$317.9204012 + 354.6421808i$
	1	$6.342841226 - 24.95915206i$	3		$540.1554148 + 583.4781202i$
	2	$12.94011467 - 41.62859430i$	5	0	$6.400241451 + 55.35392070i$
3	0	$0.3142845589 + 12.96051935i$	1		$301.0022385 + 362.7954640i$
	1	$38.84486810 - 53.32057992i$	2		$697.9325995 + 777.2504936i$
	2	$93.40199364 - 111.3133472i$	3		$1185.464821 + 1279.187469i$
	3	$160.5504266 - 181.4993689i$	4		$1749.721283 + 1855.936081i$

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TABLE V. Quasinormal modes of scalar perturbations for $\alpha = 2$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$28.98497167 - 36.65093824i$	4	0	$14.72079936 - 30.28374333i$
	1	$103.8128126 - 126.3594331i$	1		$47.53015516 - 80.58865151i$
2	0	$29.20727668 - 38.11077359i$	2		$100.4540639 - 146.2752812i$
	1	$106.1438323 - 128.6137680i$	3		$169.1566913 - 224.7764688i$
	2	$224.2314700 - 254.7665296i$	5	0	$.5248454700 + 28.40271014i$
3	0	$26.35460874 - 37.16001393i$	1		$45.08904231 - 85.15794051i$
	1	$95.10645314 - 118.6073793i$	2		$116.4495623 - 166.1450658i$
	2	$200.6518286 - 232.2842502i$	3		$205.7296676 - 263.8287746i$
	3	$333.1889473 - 371.1040372i$	4		$309.9693513 - 375.5150908i$

 TABLE VI. Quasinormal modes of massive scalar perturbations for $\alpha = 0.677$ and $\mu = 0.1M$ with $M = 1$.

l	n	ω_{nl}	l	n	ω_{nl}
1	0	$29.00936098 - 36.67332200i$	4	0	$14.74857813 - 30.30037176i$
	1	$103.9107237 - 126.4501766i$	1		$47.63840864 - 80.66160380i$
2	0	$29.23683246 - 38.13734008i$	2		$100.6803470 - 146.4444515i$
	1	$106.2610002 - 128.7222323i$	3		$169.5250565 - 225.0711954i$
	2	$224.4751039 - 254.9976125i$	5	0	$0.5217309759 + 28.39953701i$
3	0	$26.38849620 - 37.18864399i$	1		$45.02864470 - 85.11181068i$
	1	$95.24043132 - 118.7279173i$	2		$116.3098696 - 166.0264653i$
	2	$200.9301766 - 232.5433005i$	3		$205.4963229 - 263.6210918i$
	3	$333.6438082 - 371.5352605i$	4		$309.6292522 - 375.2043911i$

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